Maximaand Minima

4 Marks Questions

1. The sides of an equilateral triangle are increasing at the rate of 2cm/s. Find the rate at which the area increases, when the side is 10 cm. **All India 2014C**

Let the side of triangle be a.

$$\frac{da}{dt} = 2 \text{ cm/s} \qquad \text{[given]} \quad \textbf{(1)}$$

Now, area of equilateral triangle having side a is given by

$$A = \frac{\sqrt{3}a^2}{4} \qquad \textbf{(1)}$$

On differentiating w.r.t.t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot (2a) \frac{da}{dt} \qquad \textbf{(1)}$$

On putting $\frac{da}{dt} = 2$ cm/s and $a = 10$ cm, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2 \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{s} \qquad \textbf{(1)}$$

2. The sum of the perimeters of a circle and square is k, where k is some constant. Prove that the sum of their areas is least, when the side of the square is double the radius of the circle. **Delhi 2014C; All India 2008**

Let r be the radius of circle and x be the side of a square. Then, given that

Perimeter of square + Circumference of circle

$= k$ (constant) **(1)**

i.e. $4x + 2\pi r = k$

$$\Rightarrow \quad x = \frac{k - 2\pi r}{4} \quad \text{...(i)} \ \mathbf{(1)}$$

Let A denotes the sum of their areas.

$$\therefore \quad A = x^2 + \pi r^2 \quad \text{...(ii)}$$

$$\left[\begin{array}{l} \because \text{ area of a square} = (\text{Side})^2 \\ \text{and area of circle } = \pi r^2 \end{array}\right]$$

On putting the value of x from Eq. (i) in Eq.(ii), we get

$$A = \left(\frac{k - 2\pi r}{4}\right)^2 + \pi r^2$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = 2\left(\frac{k - 2\pi r}{4}\right)\left(-\frac{2\pi}{4}\right) + 2\pi r$$

$$= -\frac{\pi}{4}(k - 2\pi r) + 2\pi r \quad \mathbf{(1)}$$

For maxima and minima, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad -\frac{\pi}{4}(k - 2\pi r) + 2\pi r = 0$$

$$\Rightarrow \quad -\frac{\pi}{4}k + \frac{\pi^2 r}{2} + 2\pi r = 0$$

$$\Rightarrow \quad -\frac{r\pi}{2}(\pi + 4) = \frac{\pi}{4}k$$

$$\Rightarrow \quad r = \frac{k}{2\pi + 8} \qquad \text{...(iii)}$$

Now, $\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right) = \frac{d}{dr}\left[2\pi r - \frac{\pi}{4}(k - 2\pi r)\right]$

$$= 2\pi + \frac{2\pi^2}{4} = 2\pi + \frac{\pi^2}{2} > 0$$

$\because \quad \frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum.

From Eq. (iii), we get

$$r = \frac{k}{2\pi + 8}$$

$\Rightarrow \quad 2r\pi + 8r = k$

$\Rightarrow \quad 2r\pi + 8r = 4x + 2\pi r \qquad [\because k = 4x + 2\pi r]$

$\Rightarrow \quad 8r = 4x$ or $x = 2r$

i.e. Side of square = Double the radius of circle

Hence, sum of area of a circle and a square is least, when side of square is equal to diameter of circle or double the radius of circle. **(1)**

6 Marks Questions

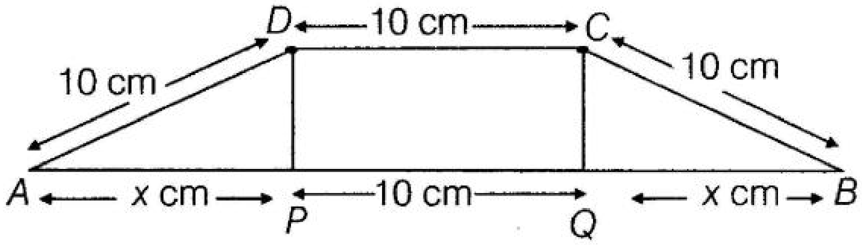
3. If the length of three sides of a trapezium other than the base are each equal to 10 cm, then find the area of the trapezium, when it is maximum. **All India 2014C, 2010; Delhi 2013C**

Let $ABCD$ be the given trapezium in which $AD = BC = CD = 10$ cm.

Let $\quad AP = x\,\text{cm}$

$\because \quad \Delta APD \cong \Delta BQC$

$\therefore \quad QB = x$ cm **(1)**



In ΔAPD,

$$DP = \sqrt{10^2 - x^2} \text{ [by Pythagoras theorem]}$$

Now, area of trapezium,

$$A = \frac{1}{2} \times \text{(Sum of parallel sides)} \times \text{Height}$$

$$= \frac{1}{2} \times (2x + 10 + 10) \times \sqrt{100 - x^2}$$

$$= (x + 10)\sqrt{100 - x^2} \quad \text{... (i)}$$ **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dA}{dx} = (x + 10)\frac{(-2x)}{2\sqrt{100 - x^2}} + \sqrt{100 - x^2}$$

$$= \frac{-x^2 - 10x + 100 - x^2}{\sqrt{100 - x^2}}$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} \quad \text{...(ii)}$$ **(1)**

For maximum, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} = 0$$

$\Rightarrow \quad 2(x + 10)(x - 5) = 0 \Rightarrow x = 5$ or -10 **(1)**

Since, x represents distance, it cannot be negative. So, we take $x = 5$.

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{\left[\begin{array}{l}\sqrt{100-x^2}(-4x-10) \\ -(-2x^2-10x+100)\left(\frac{-2x}{2\sqrt{100-x^2}}\right)\end{array}\right]}{(\sqrt{100-x^2})^2}$$

[by quotient rule]

$$= \frac{\left[\begin{array}{l}(100-x^2)(-4x-10) \\ \quad -(-2x^2-10x+100)(-x)\end{array}\right]}{(100-x^2)^{3/2}}$$

$$= \frac{\left[\begin{array}{l}-400x-1000+4x^3+10x^2 \\ \quad +(-2x^3-10x^2+100x)\end{array}\right]}{(100-x^2)^{3/2}}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{3/2}} \quad \textbf{(1)}$$

At $x = 5$, $\frac{d^2A}{dx^2} = \frac{2(5)^3 - 300(5) - 1000}{[100-(5)^2]^{3/2}}$

$$= \frac{250-1500-1000}{(100-25)^{3/2}} = \frac{-2250}{75\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and maximum value is

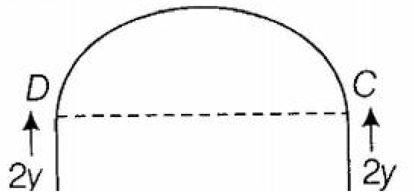
$A_{max} = (5+10)\sqrt{100-(5)^2}$ [put $x = 5$ in Eq. (i)]

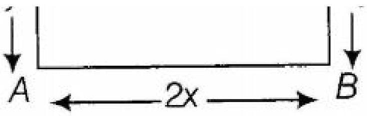
$= 15\sqrt{100-25} = 15\sqrt{75} = 75\sqrt{3}\text{ cm}^2$ **(1)**

4. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10m. Find the dimensions of the window to admit maximum light through the whole opening.

Foreign 2014; All India 2011

Let the dimensions of the window be $2x$ and $2y$, i.e. $AB = 2x$ and $BC = 2y$. Again, let P denotes the perimeter of the window and A denotes its area.





Given , $P = 10\,\text{m}$

To find dimensions of the window, so that maximum light pass through the whole opening.

From the figure, we can see that perimeter of the window is given by

$$P = 2x + 2y + 2y + \pi x$$

[$\because$ in the semi-circle, $2x$ is the diameter, so radius is x]

$$\Rightarrow \quad P = 2x + 4y + \pi x$$

$$\Rightarrow \quad 2x + 4y + \pi x = 10 \qquad [\because P = 10\text{ m}]$$

$$\Rightarrow \quad y = \frac{10 - \pi x - 2x}{4} \qquad \text{...(i)} \; \textbf{(1)}$$

Also, area of the whole window is given by

$$A = (2x)(2y) + \frac{\pi x^2}{2}$$

$$\left[\because \text{area of window} = \text{area of rectangle} + \text{area of semi-circle}\right]$$

$$\Rightarrow \quad A = 4xy + \frac{\pi x^2}{2} \qquad \text{...(ii)}$$

5.

On putting value of y from Eq. (i) in Eq. (ii), we get

$$A = 4x\left(\frac{10 - \pi x - 2x}{4}\right) + \frac{\pi x^2}{2}$$

$$\Rightarrow \quad A = 10x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$

$$\Rightarrow \quad A = 10x - 2x^2 - \frac{\pi x^2}{2} \qquad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 10 - 4x - \frac{2\pi x}{2}$$

$$= 10 - 4x - \pi x \qquad \textbf{(1)}$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 10 - 4x - \pi x = 0$$

$$\Rightarrow \quad x = \frac{10}{\pi + 4} \qquad \textbf{(1)}$$

On putting value of x in Eq. (i), we get

$$y = \frac{10 - \pi\left(\frac{10}{\pi + 4}\right) - 2\left(\frac{10}{\pi + 4}\right)}{4}$$

$$\Rightarrow \quad y = \frac{\left(10 - \frac{10\pi}{\pi + 4} - \frac{20}{\pi + 4}\right)}{4}$$

$$\Rightarrow \quad y = \frac{20}{4(\pi + 4)} = \frac{5}{\pi + 4} \qquad \textbf{(1)}$$

$$\therefore \quad x = \frac{10}{\pi + 4} \text{ and } y = \frac{5}{\pi + 4}$$

Now, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}(10 - 4x - \pi x)$

$$= -4 - \pi < 0$$

$\therefore$ A is maximum.

Hence, maximum light passes through the window.

And dimensions of window are

$$2x = \frac{20}{\pi + 4} \text{ and } 2y = \frac{10}{\pi + 4} \qquad \textbf{(1)}$$

5. Find the point p on the curve $y^2 = 4ax$, which is nearest to the point $(11\,a, 0)$.

All India 2014C

Let the point on $y^2 = 4ax$ be (x_1, y_1). Then,

$y_1^2 = 4ax_1$. ...(i)

Distance between (x_1, y_1) and $(11a, 0)$ is given by

$$D = \sqrt{(x_1 - 11a)^2 + (y_1 - 0)^2}$$

$$= \sqrt{(x_1 - 11a)^2 + y_1^2}$$

$$= \sqrt{(x_1 - 11a)^2 + 4ax_1} \text{ [from Eq. (i)]} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x_1, we get

$$\frac{dD}{dx_1} = \frac{1}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}} \cdot [2(x_1 - 11a) + 4ax_1] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dD}{dx_1} = \frac{2x_1 - 22a + 4a}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}} = 0$$

$$\Rightarrow \quad \frac{dD}{dx_1} = \frac{x_1 - 9a}{\sqrt{(x_1 - 11a^2) + 49x_1}}$$

$\Rightarrow$ Put $\quad \dfrac{dD}{dx_1} = 0 \quad \Rightarrow x_1 - 9a = 0$

$\Rightarrow \quad x_1 = 9a$

If $x_1 = 9a$, then $y_1^2 = 36a^2$

$\Rightarrow \quad y_1 = \pm 6a$ **(1)**

Hence, required points are $(9a, 6a)$ and $(9a, -6a)$.

Now, $\quad \dfrac{d^2D}{dx_1^2} = \dfrac{d}{dx_1}\left[\dfrac{dD}{dx_1}\right]$ **(1)**

$$= \frac{d}{dx_1}\left(\frac{x_1 - 9a}{\sqrt{(x_1 - 11a)^2 + 49x_1}}\right)$$

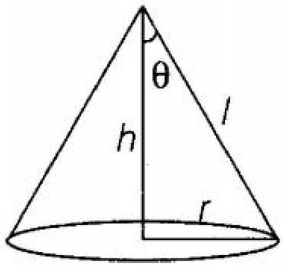
$$= \frac{\sqrt{(x_1 - 11a)^2 + 4ax_1} - (x_1 - 9a)\cdot\dfrac{1\cdot[2(x_1 - 11a) + 4a]}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}}}{(x_1 - 11a)^2 + 4ax_1}$$

At $(9a, 6a)$, $\dfrac{d^2D}{dx_1^2} > 0$ **(1)**

So, at $(9a, 6a)$, D is minimum.

Hence, required point is $(9a, 6a)$. **(1)**

Let r be the radius of the base, h be the height, V be the volume, S be the surface area of the cone and θ be the semi-vertical angle.



(1)

Then, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \quad 3V = \pi r^2 h$

$\Rightarrow \quad 9V^2 = \pi^2 r^4 h^2$ [squaring on both sides]

$\Rightarrow \quad h^2 = \frac{9V^2}{\pi^2 r^4}$ ...(i) **(1)**

and curved surface area, $S = \pi r l$

$\Rightarrow \quad S = \pi r \sqrt{r^2 + h^2}$ $[\because l = \sqrt{h^2 + r^2}]$

$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2)$

[squaring on both sides]

$\Rightarrow \quad S^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$ [from Eq. (i)]

$\Rightarrow \quad S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$ ...(ii)

When S is least, then S^2 is also least. **(1)**

Now, $\frac{d}{dr}(S^2) = -\frac{18V^2}{r^3} + 4\pi^2 r^3$ ...(iii)

For maxima or minima, put $\frac{d}{dr}(S^2) = 0$ **(1)**

$$\Rightarrow \quad -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0 \Rightarrow 18V^2 = 4\pi^2 r^6$$

$$\Rightarrow \quad 9V^2 = 2\pi^2 r^6 \quad \text{...(iv)}$$ **(1)**

Again, on differentiating Eq. (iii) w.r.t. r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

At $\quad 9V^2 = 2\pi^2 r^6,$

$$\frac{d^2}{dr^2}(S^2) = \frac{54}{r^4}\left(\frac{2\pi^2 r^6}{9}\right) + 12\pi^2 r^2$$

$$= \frac{12\pi^2 r^6}{r^4} + 12\pi^2 r^2 = 24\pi^2 r^2 > 0$$

So, S^2 and S is minimum, when

$$9V^2 = 2\pi^2 r^6$$ **(1)**

On putting $9V^2 = 2\pi^2 r^6$ in Eq. (i), we get

$$2\pi^2 r^6 = \pi^2 r^4 h^2$$

$$\Rightarrow \quad 2r^2 = h^2 \Rightarrow h = \sqrt{2}\, r$$

$$\Rightarrow \quad \frac{h}{r} = \sqrt{2} \Rightarrow \cot\theta = \sqrt{2}$$

[from the figure, $\cot\theta = \frac{h}{r}$]

$$\Rightarrow \quad \theta = \cot^{-1}\sqrt{2}$$

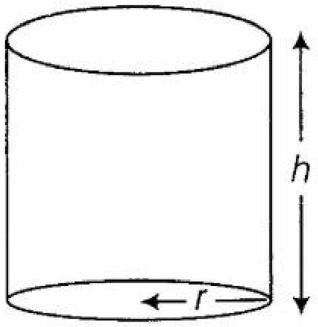
Hence, the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **(1)**

Hence proved.

7. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area.

Delhi 2014

Let r cm be the radius of base and h cm be the height of the cylindrical can. Let its volume be V and S be its total surface area. Then,



Can

(1)

$$V = 128\pi \text{ cm}^3 \quad \text{[given]}$$

$$\Rightarrow \quad \pi r^2 h = 128\pi$$

$$\Rightarrow \quad h = \frac{128}{r^2} \quad \text{...(i)}$$

Also, $$S = 2\pi r^2 + 2\pi r h \quad \text{...(ii)}$$

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right) \quad \text{[using Eq.(i)]}$$

(1)

$$\Rightarrow \quad S = 2\pi r^2 + \frac{256\pi}{r} \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \text{...(iv)}$$ **(1)**

For maxima or minima, put $\frac{dS}{dr} = 0$.

$$\Rightarrow \quad 4\pi r = \frac{256\pi}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{256}{4}$$

$$\Rightarrow \quad r^3 = 64$$

Taking cube root on both sides, we get

$$r = (64)^{1/3}$$

$$\Rightarrow \quad r = 4 \text{ cm} \qquad \textbf{(1)}$$

Again,on differentiating Eq. (iv) w.r.t. r, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

At $r = 4$,

$$\frac{d^2S}{dr^2} = \frac{512\pi}{64} + 4\pi$$

$$= 8\pi + 4\pi = 12\pi > 0 \qquad \textbf{(1)}$$

Thus, $\frac{d^2s}{dr^2} > 0$ at $r = 4$, so the surface area is minimum, when the radius of cylinder is 4 cm.

On putting value of r in Eq. (i), we get

$$h = \frac{128}{(4)^2}$$

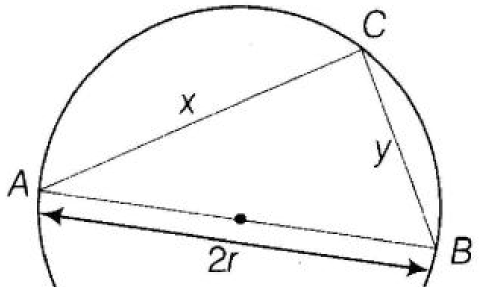
$$= \frac{128}{16} = 8 \text{ cm}$$

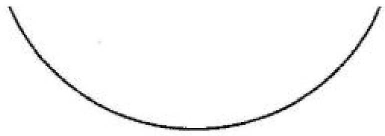
Hence, for the minimum surface area of can, the dimensions of the can are $r = 4$ cm and $h = 8$ cm. **(1)**

8. *AB* is a diameter of a circle and *C* is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. **All India 2014C**

Let the side of ΔABC be x and y and r be the radius of circle.

Also, $\angle C = 90°$ [$\because$ angle made in semi-circle]





(1)

In the ΔABC, we have

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$\Rightarrow \quad (2r)^2 = (x)^2 + (y)^2$$

$$\Rightarrow \quad 4r^2 = x^2 + y^2 \quad \text{...(i)} \mathbf{(1)}$$

Area of ΔABC $(A) = \frac{1}{2} x \cdot y$

On squaring both sides, we get

$$A^2 = \frac{1}{4} x^2 y^2$$

Let $\quad A^2 = S$

Then, $\quad S = \frac{1}{4} x^2 y^2$

From the Eq. (i), substituting the value of y^2, we get

$$S = \frac{1}{4} x^2 (4r^2 - x^2)$$

$$\Rightarrow \quad S = \frac{1}{4} (4x^2 r^2 - x^4) \quad \mathbf{(1)}$$

On differentiating w.r.t.x, we get

$$\frac{dS}{dx} = \frac{1}{4} (8r^2 x - 4x^3)$$

For maximum and minimum, put $\frac{dS}{dx} = 0$.

$$\Rightarrow \quad 0 = \frac{1}{4} (8r^2 x - 4x^3)$$

$$\Rightarrow \quad 8r^2 x = 4x^3 \Rightarrow 8r^2 = 4x^2$$

$$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2} r$$

Then, from the Eq.(i), we get

$$y^2 = 4r^2 - 2r^2 = 2r^2$$

$$\Rightarrow \quad y = \sqrt{2} r \quad \mathbf{(1)}$$

i.e. $x = y$, so triangle is isosceles.

Also, $\frac{d^2S}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}(8r^2x - 4x^3)\right]$ **(1)**

$$= \frac{1}{4}[8r^2 - 12x^2] = 2r^2 - 3x^2$$

At $x = \sqrt{2r}, \frac{d^2S}{dx^2} = 2r^2 - 3x^2 < 0$

$\Rightarrow$ Area is maximum.

Hence, area is maximum, when triangle is isosceles. **(1)**

9. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also, show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. **All India 2014**

Let R be the radius and h be the height of the cone, which inscribed in a sphere of radius r.

$\therefore \quad OA = h - r$

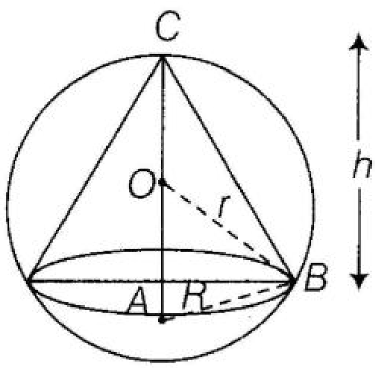
In ΔOAB, by Pythagoras theorem, we have

$$r^2 = R^2 + (h-r)^2$$

$$\Rightarrow \quad r^2 = R^2 + h^2 + r^2 - 2rh$$

$$\Rightarrow \quad R^2 = 2rh - h^2 \quad \text{...(i)}\mathbf{(1)}$$

The volume of sphere $= \frac{4}{3}\pi r^3$



and the volume V of the cone,

$$V = \frac{1}{3}\pi R^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi h(2rh - h^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(2rh^2 - h^3) \quad \text{...(ii)}$$

On differentiating Eq. (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2) \quad \text{...(iii)}\mathbf{(1)}$$

For maximum or minimum, put $\frac{dV}{dh} = 0$.

$$\Rightarrow \quad \frac{1}{3}\pi(4rh - 3h^2) = 0$$

$$\Rightarrow \quad 4rh = 3h^2 \Rightarrow 4r = 3h$$

$$\Rightarrow \quad h = \frac{4r}{3} \quad [\because h \neq 0]\ \mathbf{(1)}$$

Again, on differentiating Eq. (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$$

At $h = \frac{4r}{3}$, $\left[\frac{d^2V}{dh^2}\right]_{h=\frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r) = -\frac{4r\pi}{3} < 0$$

$\Rightarrow V$ is maximum at $h = \frac{4r}{3}$. **(1)**

On substituting the value of h in Eq. (ii), we get

$$V = \frac{1}{3}\pi\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right]$$

$$= \frac{\pi}{3}\left[\frac{32}{9}r^3 - \frac{64}{27}r^3\right] = \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

$$= \frac{\pi}{3}r^3\left[\frac{96-64}{27}\right] = \frac{\pi}{3}r^3\left(\frac{32}{27}\right)$$

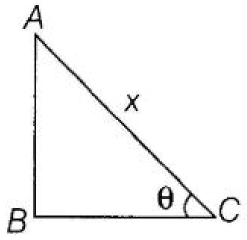
$= \frac{8}{27} \times \left(\frac{4}{3}\pi r^3\right) = \frac{8}{27} \times$ (Volume of sphere)**(1)**

Hence, maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

10. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, then show that the area of the triangle is maximum, when the angle between them is $60°$. All India 2014

Let ABC be a right angled triangle.

Given, $AC + BC = \text{constant} = k$ ...(i)



(1)

Let $\angle ACB = \theta$ and $AC = x$.

Then, $BC = x\cos\theta$ and $AB = x\sin\theta$

Let y be the area of ΔABC.

Then, $y = \frac{1}{2} BC \cdot AB = \frac{1}{2} x\cos\theta \cdot x\sin\theta$

$$= \frac{1}{2} x^2 \sin\theta\cos\theta \quad \text{...(ii)}$$

From Eq. (i), $x + x\cos\theta = k$

$$\Rightarrow \quad x = \frac{k}{1+\cos\theta} \quad \text{...(iii)}\textbf{(1)}$$

On putting the value of x in Eq. (ii), we get

$$y = \frac{k^2}{2} \frac{\sin\theta\cos\theta}{(1+\cos\theta)^2}$$

On differentiating w.r.t. θ, we get

$$\frac{dy}{dx} = \frac{k^2}{2} \frac{\begin{bmatrix}(1+\cos\theta)^2(\cos^2\theta - \sin^2\theta) \\ -\sin\theta\cos\theta\, 2\,(1+\cos\theta)(-\sin\theta)\end{bmatrix}}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{\begin{bmatrix}(1+\cos\theta)[(1+\cos\theta)(\cos^2\theta - \sin^2\theta) \\ + 2\sin^2\theta\cos\theta]\end{bmatrix}}{(1+\cos\theta)^4}$$

(1)

$$= \frac{k^2}{2} \frac{\begin{bmatrix}\cos^2\theta - \sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta \\ + 2\cos\theta\sin^2\theta\end{bmatrix}}{(1+\cos\theta)^3}$$

$$= \frac{k^2(2\cos^2\theta - 1 + \cos^3\theta + \cos\theta\sin^2\theta)}{2(1+\cos\theta)^3}$$

$$[\because \sin^2\theta = 1 - \cos^2\theta]$$

$$k^2[2\cos^2\theta - 1 + \cos\theta(\cos^2\theta + \sin^2\theta)]$$

$$= \frac{}{2(1+\cos\theta)^3}$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1)$$

$[\because \cos^2\theta + \sin^2\theta = 1]$ **(1)**

Since, $0 < \theta < \frac{\pi}{2} \Rightarrow \frac{k^2}{2(1+\cos\theta)^3} > 0$

$\therefore$ Sign of $\frac{dy}{dx}$ will be depend on $2\cos^2\theta + \cos\theta - 1$.

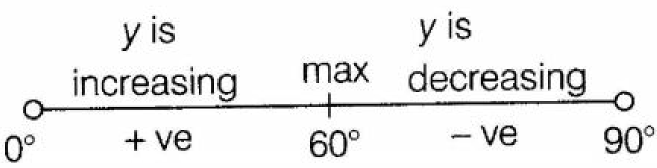
Now, $\quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad [\because \cos\theta \neq -1]$

$\Rightarrow \quad \theta = 60^\circ \quad [\because 0 < \theta < 90^\circ]$ **(1)**

Then, sign scheme for $\frac{dy}{dx}$, i.e. for $(2\cos^2\theta + \cos\theta - 1)$ is



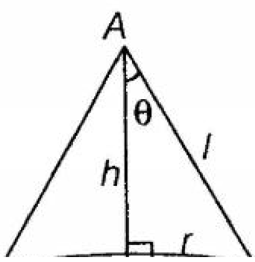
Thus, y has maximum value, when $\theta = 60^\circ$. **(1)**

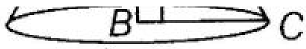
11. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1} 1/\sqrt{3}$. **Delhi 2014**

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$.

Let r, h and l be the radius, height and the slant height of the cone, respectively.





(1)

The slant height of the cone is given, i.e. consider as constant.

Now, in ΔABC, $r = l \sin\theta$ and $h = l \cos\theta$

Let V be the volume of the cone.

Then, $V = \frac{\pi}{3} r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi (l^2 \sin^2\theta)(l \cos\theta)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi l^3 \sin^2\theta \cos\theta \qquad \textbf{(1)}$$

On differentiating w.r.t. θ two times, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$

$$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta) \qquad \textbf{(1)}$$

For maxima or minima, put $\frac{dV}{d\theta} = 0$.

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta \quad \Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2} \quad \Rightarrow \theta = \tan^{-1}\sqrt{2}$ **(1)**

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$

$$\Rightarrow \quad \sin^2\theta = 2\cos^2\theta$$

Then, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$$= -4\pi l^3\cos^3\theta < 0, \text{ for } \theta \in \left(0, \frac{\pi}{2}\right) \qquad \textbf{(1)}$$

$\therefore V$ is maximum, when $\theta = \tan^{-1}\sqrt{2}$ or

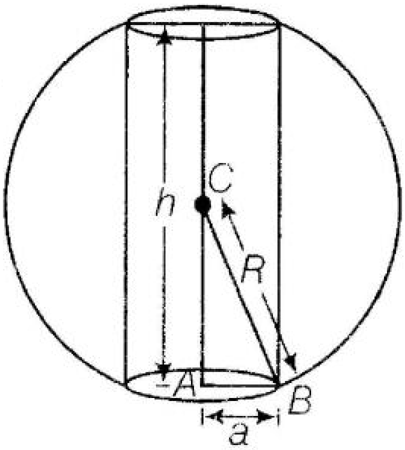
$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}. \qquad \left[\because \cos\theta = \frac{1}{\sqrt{3}}\right]$$

Hence, for given slant height, the semi-vertical angle of the cone of maximum volume is $\cos^{-1}\frac{1}{\sqrt{3}}$. **(1)**

12. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume.

All India 2014,2012C,2011; Delhi 2013

Let h be the height and a be the radius of base of cylinder inscribed in the given sphere of radius (R).



In ΔABC,

$$AB^2 + AC^2 = BC^2 \text{ [by Pythagoras theorem]}$$

$$\Rightarrow \quad a^2 + \left(\frac{h}{2}\right)^2 = R^2 \quad \Rightarrow \quad a^2 = R^2 - \frac{h^2}{4} \qquad \textbf{(1)}$$

Volume of cylinder, $V = \pi a^2 h$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \frac{\pi}{4}(4R^2h - h^3) \qquad \textbf{(1)}$$

On differentiating both sides two times w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$$

and $$\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3\pi h}{2} \quad \text{...(i)} \ \textbf{(1)}$$

For maxima or minima, put $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{\pi}{4}(4R^2 - 3h^2) = 0$$

$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2}{\sqrt{3}}R \quad \textbf{(1)}$$

[$\because$ height is always positive, so we do not take '–' sign]

On substituting value of h in Eq. (i), we get

$$\frac{d^2V}{dh^2} = \frac{-3\pi}{2} \cdot \frac{2}{\sqrt{3}}R = -\sqrt{3}\pi R < 0 \quad \textbf{(1)}$$

$\Rightarrow$ V is maximum.

Hence, required height of cylinder is $\frac{2R}{\sqrt{3}}$.

Now, maximum volume of cylinder

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi \frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}\cdot\frac{4}{3}R^2\right)$$

$$\left[\text{put } h = \frac{2}{\sqrt{3}}R\right]$$

$$= \frac{2\pi R}{\sqrt{3}} \frac{(3R^2 - R^2)}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cu units} \quad \textbf{(1)}$$

Hence proved.

13. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base. **Foreign 2014; Delhi 2011C, 2009**

Let r be the radius, h be the height, V be the volume and S be the total surface area of a right circular cylinder which is open at the top.

Now, given that

$$V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2} \qquad \ldots\text{(i)} \ \textbf{(1)}$$

Also, we know that, total surface area S is given by

$$S = 2\pi rh + \pi r^2$$

[$\because$ cylinder is open at the top, therefore
S = curved surface area of cylinder
+ area of base]

$$\Rightarrow \quad S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2$$

$\left[\text{putting } h = \frac{V}{\pi r^2}, \text{ from Eq. (i)}\right]$

$$\Rightarrow \quad S = \frac{2V}{r} + \pi r^2 \qquad \textbf{(1)}$$

On differentiating w.r.t. r, we get

$$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow -\frac{2V}{r^2} + 2\pi r = 0$$

$$\Rightarrow \quad V = \pi r^3$$

$$\Rightarrow \quad \pi r^2 h = \pi r^3 \qquad [\because V = \pi r^2 h, \text{ given}]$$

$$\Rightarrow \quad h = r \qquad \textbf{(1)}$$

Also, $$\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(\frac{-2V}{r^2} + 2\pi r\right)$$

$$\Rightarrow \quad \frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi \qquad \textbf{(1)}$$

On putting $r = h$, we get

$$\left[\frac{d^2S}{dr^2}\right]_{r=h} = \frac{4V}{h^3} + 2\pi > 0 \text{ as } h > 0 \qquad \textbf{(1)}$$

Then, $\frac{d^2S}{dr^2} > 0 \Rightarrow S$ is minimum.

Hence, S is minimum, when $h = r$, i.e. when height of cylinder is equal to radius of the base.

(1)

14. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

All India 2013

Let $ABCD$ be a rectangle having area A inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

Let the coordinates of A be (α, β).

Then, coordinates of $B = (\alpha_1 - \beta)$

$C = (\alpha_1 - \beta)$

$D = (\alpha_1 - \beta)$ **(1)**

Area, A = Length$\times$ Breadth $= 2\alpha \times 2\beta$

$$\Rightarrow \quad A = 4\alpha\beta$$

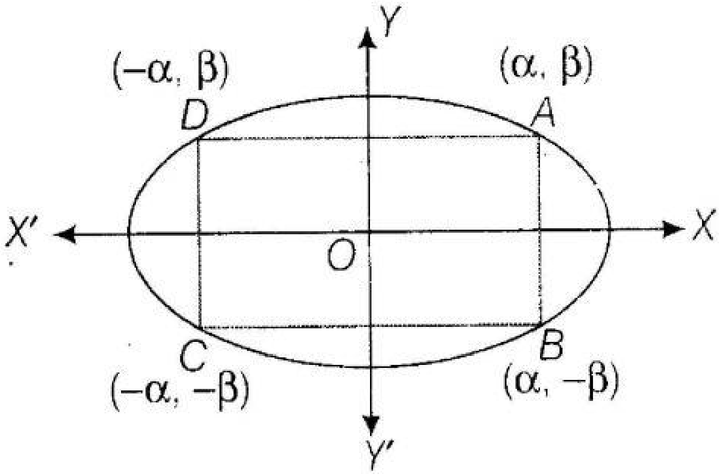
$$\Rightarrow \quad A = 4\alpha . \sqrt{b^2\left(1 - \frac{\alpha^2}{a^2}\right)} \qquad \textbf{(1)}$$

$$\left[\begin{array}{r} \because (\alpha, \beta) \text{ lies on ellipse} \\ \therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1, \text{ i.e. } \beta = \sqrt{b^2\left(\frac{1-\alpha^2}{a^2}\right)} \end{array}\right]$$

$$\Rightarrow \quad A^2 = 16\alpha^2\left\{b^2\left(\frac{1-\alpha^2}{a^2}\right)\right\}$$

[on squaring

$$\Rightarrow \quad A^2 = \frac{16b^2}{a^2}(a^2\alpha^2 - \alpha^4)$$



On differentiating w.r.t. α, we get

$$\frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$$

For maximum or minimum value, put

$$\frac{dA^2}{d\alpha} = 0$$

$$\Rightarrow \quad 2a^2\alpha - 4\alpha^3 = 0$$

$$\Rightarrow \quad 2\alpha(a^2 - 2\alpha^2) = 0$$

$$\Rightarrow \quad \alpha = 0, \alpha = \frac{a}{\sqrt{2}} \qquad \textbf{(1)}$$

Again, $\dfrac{d^2(A^2)}{d\alpha^2} = \dfrac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

At $\quad \alpha = \dfrac{a^2}{\sqrt{2}}, \left(\dfrac{d^2A^2}{d\alpha^2}\right)_{\alpha = \frac{a}{\sqrt{2}}}$

$$= \frac{16b^2}{a^2}\left(2a^2 - 12 \times \frac{a^2}{2}\right) = 0 \qquad \textbf{(1)}$$

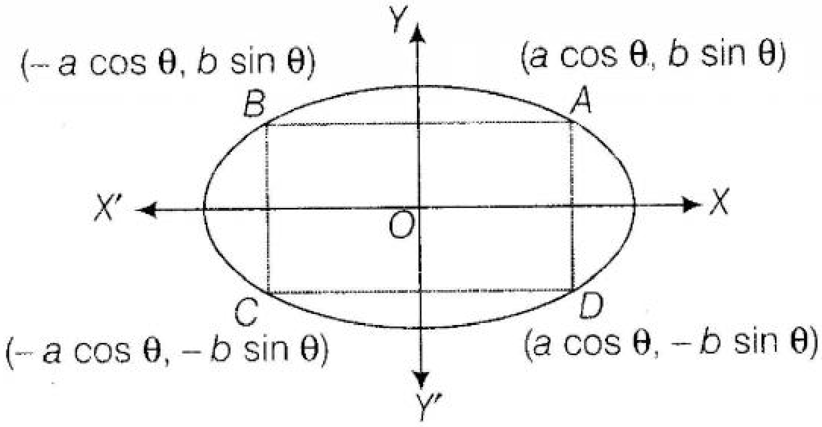
$\Rightarrow$ For $\alpha = \dfrac{a}{\sqrt{2}}, A^2$ i.e. A is maximum.

Then, from Eq. (i), we get $\beta = \dfrac{b}{\sqrt{2}}$

$\therefore$ Greatest area $= 4\alpha\beta = 4 \cdot \frac{\alpha}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = 2ab$ **(1)**

Alternate Method

Let $A\,(a\cos\theta, b\sin\theta)$ be the parametric coordinates of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where θ is the eccentric angle. **(1)**



(1)

Here, length of $AB = a\cos\theta + a\cos\theta$

$= 2a\cos\theta$

and length of $AD = b\sin\theta + b\sin\theta$

$= 2b\sin\theta$ **(1)**

Now, area of rectangle $ABCD$

$= (2a\cos\theta)(2b\sin\theta)$

$= 2ab\,(2\sin\theta \cdot \cos\theta)$

$= 2ab\sin 2\theta$ **(1½)**

Here, area of rectangle $ABCD$ is greatest, when $\sin 2\theta$ is greatest.

i.e. $\sin 2\theta = 1 = \sin 90^\circ$

$\Rightarrow$ $2\theta = 90^\circ \Rightarrow \theta = 45^\circ$ **(1½)**

Hence, area of greatest rectangle is equal to $2ab$, when eccentric angle of an ellipse is 45°.

15. Show that the height of a closed right circular cylinder of given surface and maximum volume is equal to diameter of base. **Delhi 2012**

Here, use the relations, total surface area of cylinder, $S = 2\pi r^2 + 2\pi rh$ and volume of cylinder $V = \pi r^2 h$. To make a relation between V and S and differentiate with respect to r. Then, put $\frac{dV}{dr} = 0$. And determine r and then check $\frac{d^2V}{dr^2}$, if it is negative, then maxima and if it is positive, then minima.

Let S be the surface area, V be the volume, h be the height and r be the radius of base of the right circular cylinder. We know that,

Surface area of right circular cylinder,

$$S = 2\pi r^2 + 2\pi rh \quad \ldots(i)$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad \ldots(ii)$$ **(1)**

Also, volume of right circular cylinder is given by

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad V = \frac{rS - 2\pi r^3}{2}$$ **(1)**

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2}$$ **(1)**

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S - 6\pi r^2}{2} = 0 \Rightarrow S = 6\pi r^2$$ **(1)**

From Eq. (ii), we get

$$h = \frac{6\pi r^2 - \pi r^2}{2\pi r} \quad \Rightarrow \quad h = 2r$$

$\therefore$ Height = Diameter of the base **(1)**

Also $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 6\pi r^2}{2}\right)$

Also, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\quad 2 \quad \right)$

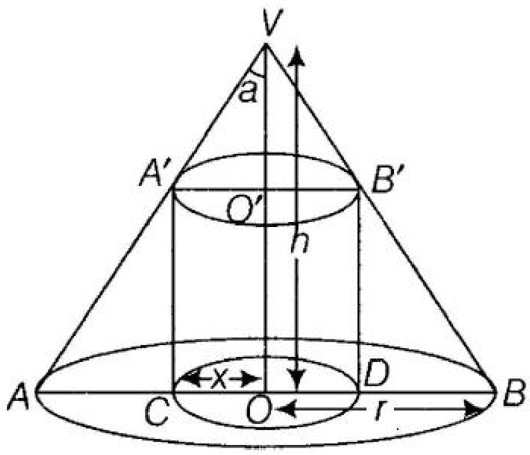
$$= -6\pi r < 0$$

$\Rightarrow V$ is maximum. **(1)**

Hence, V is maximum at $h = 2r$.

16. Prove that radius of right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. All India 2012

Let VAB be the cone of base radius r and height h. And radius of base of the inscribed cylinder be x.



(1)

Now, we observe that

$$\Delta VOB \sim \Delta B'DB \Rightarrow \frac{VO}{B'D} = \frac{OB}{DB}$$

$$\Rightarrow \quad \frac{h}{B'D} = \frac{r}{r-x} \quad \Rightarrow B'D = \frac{h(r-x)}{r}$$ **(1)**

Let C be the curved surface area of cylinder. Then,

$$C = 2\pi\,(OC)\,(B'D)$$

$$\Rightarrow \quad C = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$ **(1)**

On differentiating w.r.t. x, we get

$$\frac{dC}{dx} = \frac{2\pi h}{r}(r - 2x)$$

For maxima and minima, put $\frac{dC}{dx} = 0$

$$\Rightarrow \frac{2\pi h}{r}(r - 2x) = 0 \quad \Rightarrow \quad r - 2x = 0$$

r

$$\Rightarrow \quad r = 2x \Rightarrow x = \frac{r}{2} \quad \textbf{(1)}$$

Hence, radius of cylinder is half of that of cone.

Also, $\frac{d^2C}{dx^2} = \frac{d}{dx}\left[\frac{2\pi h(r-2x)}{r}\right]$

$$= \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r} < 0 \text{ as } h, r > 0 \ \textbf{(1)}$$

$\Rightarrow C$ is maximum or greatest.

Hence, C is greatest at $x = \frac{r}{2}$. **(1)**

Hence proved.

17. An open box with a square base is to be made out of a given quantity of cardboard of area C^2 sq units. Show that the maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units. All India 2012

Let the dimensions of the box be x and y. Also, let V denotes its volume and S denotes its total surface area.

Now, $S = x^2 + 4xy$ [$\because S$ = area of square base + area of the four walls]

Given, $x^2 + 4xy = C^2$

$$\Rightarrow \quad y = \frac{C^2 - x^2}{4x} \quad \text{...(i)} \ \textbf{(1)}$$

Also, volume of the box is given by

$$V = x^2y \Rightarrow V = x^2\left(\frac{C^2 - x^2}{4x}\right) \text{ [from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{xC^2 - x^3}{4} \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{C^2 - 3x^2}{4} \quad \textbf{(1)}$$

For maxima and minima, put $dV/dx = 0$

$$\Rightarrow \quad \frac{C^2 - 3x^2}{4} = 0 \quad \Rightarrow \quad C^2 = 3x^2$$

$$\therefore \quad x = C/\sqrt{3}$$ **(1)**

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\left(\frac{C^2 - 3x^2}{4}\right)$$

$$= \frac{-6x}{4} = \frac{-3x}{2} < 0$$

$$\because \quad \frac{d^2V}{dx^2} < 0 \Rightarrow V \text{ is maximum.}$$ **(1)**

Now, maximum volume,

$$V = \frac{xC^2 - x^3}{4} = \frac{1}{4}[xC^2 - x^3]$$

$$= \frac{1}{4}\left[\frac{C}{\sqrt{3}} \cdot C^2 - \left(\frac{C}{\sqrt{3}}\right)^3\right] \quad \left[\text{put } x = \frac{C}{\sqrt{3}}\right]$$

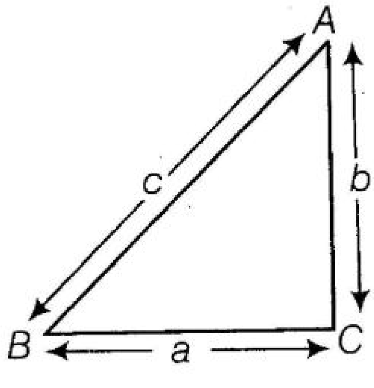
$$= \frac{1}{4}\left[\frac{C^3}{\sqrt{3}} - \frac{C^3}{3\sqrt{3}}\right] = \frac{1}{4}\left[\frac{3C^3 - C^3}{3\sqrt{3}}\right]$$

$$= \frac{1}{4} \times \frac{2C^3}{3\sqrt{3}} = \frac{C^3}{6\sqrt{3}}$$

Hence, maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units. **(1)**

18. Prove that the area of a right-angled triangle of given hypotenuse is maximum, when the triangle is isosceles. **Delhi 2012C**

Let a and b be the sides of right angled triangle.



(1/2)

From ΔABC, we have

$$c^2 = a^2 + b^2$$

Area of ΔABC $(A) = \frac{1}{2} a \cdot b = \frac{1}{2} a\sqrt{c^2 - a^2}$

$[\because b = \sqrt{c^2 - a^2}]$ **(1)**

On differentiating w.r.t. a, we get

$$\frac{dA}{da} = \frac{1}{2} \cdot 1 \cdot \sqrt{c^2 - a^2} + \frac{1}{2} \cdot a \cdot \frac{1}{2} \cdot \frac{(-2a)}{\sqrt{c^2 - a^2}}$$

$$= \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right]$$ **(1)**

For maxima and minima, $\frac{dA}{da} = 0$

$$\Rightarrow \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right] = 0$$

$$\Rightarrow c^2 - a^2 - a^2 = 0 \Rightarrow c^2 = 2a^2$$

$$\Rightarrow a = \frac{c}{\sqrt{2}}$$

Now, $$\frac{d^2A}{da^2} = \frac{1}{2}\left[\frac{-a}{\sqrt{c^2 - a^2}} - \frac{a^3}{(c^2 - a^2)^{3/2}}\right]$$

$$= -\frac{1}{2} a\left[\frac{c^2 - a^2 + a^2}{(c^2 - a^2)^{3/2}}\right]$$ **(1)**

$$= -\frac{1}{2} \frac{c^2 a}{(c^2 - a^2)^{3/2}} < 0$$ **(1½)**

$\therefore$ Area of ΔABC is maximum and

$b = \sqrt{c^2 - a^2} = \sqrt{2a^2 - a^2} = a$ **(1)**

Hence, triangle is isosceles. **Hence proved.**

19. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **HOTS; Delhi 2011**

Using the result, volume of cone, $V = \frac{1}{3}\pi r^2 h$ and curved surface area, $S = \pi r l$. Make the relation between V and S, differentiate it and simplify to get the result.

Let C denotes the curved surface area, r be the radius of base, h be the height and V be the volume of right circular cone.

To show, $\quad h = \sqrt{2}r$

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2} \quad \text{...(i)} \ \mathbf{(1)}$$

Also, the curved surface area of cone is given by $C = \pi r l$, where $l = \sqrt{r^2 + h^2}$ is the slant height of cone.

$$\therefore \quad C = \pi r \sqrt{r^2 + h^2}$$

On squaring both sides, we get

$$C^2 = \pi^2 r^2 (r^2 + h^2) \Rightarrow C^2 = \pi^2 r^4 + \pi^2 r^2 h^2$$

Let $\quad C^2 = Z$

Then, $\quad Z = \pi^2 r^4 + \pi^2 r^2 h^2 \quad \text{...(ii)}$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \left(\frac{3V}{\pi r^2}\right)^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \times \frac{9V^2}{\pi^2 r^4} \quad \mathbf{(1½)}$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \frac{9V^2}{r^2}$$

On differentiating both sides w.r.t. r, we get

$$\frac{dZ}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} \quad \mathbf{(½)}$$

For maxima and minima, put $\frac{dZ}{dr} = 0$

$$\Rightarrow \quad 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3}$$

$$r^3 \qquad\qquad r^3$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18\left(\frac{1}{3}\pi r^2 h\right)^2 \quad \left[\because V = \frac{1}{3}\pi r^2 h\right]$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \frac{1}{9}\pi^2 r^4 h^2$$

$$\Rightarrow \quad 4\pi^2 r^6 = 2\pi^2 r^4 h^2 \Rightarrow 2r^2 = h^2 \Rightarrow h = \sqrt{2}r$$

Hence, height $= \sqrt{2}$ [radius of base]**(1½)**

Also, $\dfrac{d^2Z}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dz}{dr}\right)$

$$= \frac{d}{dr}\left(4\pi^2 r^3 - \frac{18V^2}{r^3}\right) = 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

$$\therefore \frac{d^2Z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4} > 0$$

$\Rightarrow Z$ is minimum $\Rightarrow C$ is minimum.

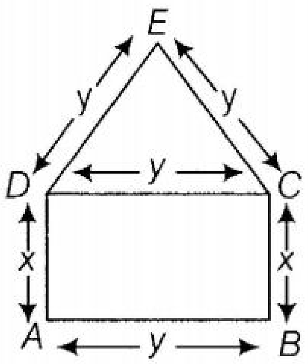
Hence, curved surface area is least, when $h = \sqrt{2}r.$ **(1½)**

NOTE If C is maximum/minimum, then C^2 is also maximum/minimum.

20. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, then find the dimensions of the rectangle that will produce the largest area of the window.

Delhi 2011

Let $ABCD$ be the rectangle which is surmounted by an equilateral ΔEDC.



Now, given that

Now, given that

Perimeter of window = 12 m

$\Rightarrow \quad 2x + 3y + y = 12$

$\therefore \quad x = 6 - 2y \quad$ ...(i) **(1)**

Let A denote the combined area of the window. Then,

$$A = xy + \frac{\sqrt{3}}{4}y^2$$

[$\because$ combined area = area of rectangle + area of equilateral triangle]

$$\Rightarrow \quad A = y(6 - 2y) + \frac{\sqrt{3}}{4}y^2 \quad \textbf{(1)}$$

[$\because x = 6 - 2y$ from Eq. (i)]

$$\Rightarrow \quad A = 6y - 2y^2 + \frac{\sqrt{3}}{4}y^2$$

On differentiating w.r.t. y, we get

$$\frac{dA}{dy} = 6 - 4y + \frac{2\sqrt{3}}{4}y \quad \textbf{(1)}$$

Now, for maxima and minima, put $\frac{dA}{dy} = 0$

$$\Rightarrow \quad 6 - 4y + \frac{2\sqrt{3}}{4}y = 0$$

$$\Rightarrow \quad y\left(\frac{\sqrt{3}}{2} - 4\right) = -6 \quad \Rightarrow \quad y = \frac{12}{8 - \sqrt{3}}$$

Now, $\frac{d^2A}{dy^2} = \frac{d}{dy}\left(\frac{dA}{dy}\right) = \frac{d}{dy}\left(6 - 4y + \frac{2\sqrt{3}}{4}y\right)$

$$= -4 + \frac{2\sqrt{3}}{4} = \frac{-8 + \sqrt{3}}{2} < 0$$

$\therefore A$ is maximum. **(1)**

Now, on putting $y = \frac{12}{8 - \sqrt{3}}$ in Eq. (i), we get

$$x = 6 - 2\left(\frac{12}{8 - \sqrt{3}}\right) \Rightarrow x = \frac{48 - 6\sqrt{3} - 24}{8 - \sqrt{3}}$$

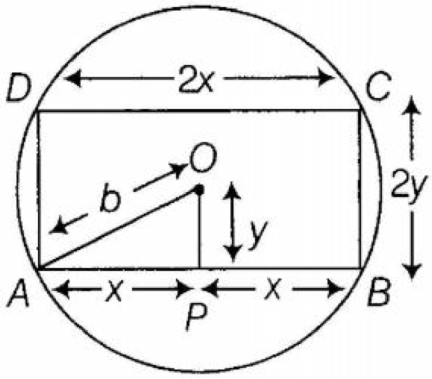
$$\Rightarrow \quad x = \frac{24 - 6\sqrt{3}}{8 - \sqrt{3}} \qquad \textbf{(1)}$$

Hence, the area of the window is largest and the dimensions of the window are

$$x = \frac{24 - 6\sqrt{3}}{8 - \sqrt{3}} \quad \text{and} \quad y = \frac{12}{8 - \sqrt{3}}. \qquad \textbf{(1)}$$

21. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. **All India 2011**

Let $ABCD$ be the rectangle which is inscribed in the fixed circle which has centre O and radius b. Let $AB = 2x$ and $BC = 2y$.



In right angled ΔOPA, by Pythagoras theorem, we have

$$OP^2 + AP^2 = OA^2$$

$$\Rightarrow \quad x^2 + y^2 = b^2 \Rightarrow y^2 = b^2 - x^2$$

$$\Rightarrow \quad y = \sqrt{b^2 - x^2} \qquad \ldots\text{(i)} \ \textbf{(1)}$$

Let A be the area of rectangle.

$$\therefore \quad A = (2x)(2y)$$

[$\because$ area of rectangle = Length $\times$ Breadth]

$$\Rightarrow \quad A = 4xy$$

$$\Rightarrow \quad A = 4x\sqrt{b^2 - x^2} \qquad [\because y = \sqrt{b^2 - x^2}]$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 4x \cdot \frac{d}{dx}\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2} \cdot \frac{d}{dx}(4x)$$

$$\frac{dA}{dx} \quad \ldots \quad -2x \quad \ldots \quad \sqrt{b^2 - x^2}$$

$$\Rightarrow \quad \frac{}{dx} = 4x \cdot \frac{}{2\sqrt{b^2 - x^2}} + \sqrt{b^2 - x^2} \cdot 4$$

$$= 4\left[\frac{b^2 - x^2 - x^2}{\sqrt{b^2 - x^2}}\right]$$

$$\Rightarrow \quad \frac{dA}{dx} = 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\therefore \quad 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) = 0$$

$$\Rightarrow \quad b^2 - 2x^2 = 0 \Rightarrow 2x^2 = b^2$$

$$\Rightarrow \quad x = \frac{b}{\sqrt{2}} \quad \textbf{(1)}$$

[$\because$ x cannot be negative]

Also, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}\left[\frac{4(b^2 - 2x^2)}{\sqrt{b^2 - x^2}}\right]$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{d}{dx}[4(b^2 - 2x^2)(b^2 - x^2)^{-1/2}]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4[-4x(b^2 - x^2)^{-1/2} + (b^2 - 2x^2) \left(-\frac{1}{2}\right)(b^2 - x^2)^{-3/2}(-2x)]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4\left[\frac{-4x}{\sqrt{b^2 - x^2}} + \frac{x(b^2 - 2x^2)}{(b^2 - x^2)^{3/2}}\right] \quad \textbf{(1)}$$

On putting $x = \frac{b}{\sqrt{2}}$, we get

$$\frac{d^2A}{dx^2} = 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{b^2 - \frac{b^2}{2}}} + \frac{\frac{b}{\sqrt{2}}\left(b^2 - 2 \times \frac{b^2}{2}\right)}{\left(b^2 - \frac{b^2}{2}\right)^{3/2}}\right]$$

$$= 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{\frac{b^2}{2}}} + 0\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = -16 < 0$$

$\because \quad \frac{d^2A}{dx^2} < 0$. So, A is maximum at $x = \frac{b}{\sqrt{2}}$. **(1)**

Now, putting $x = \frac{b}{\sqrt{2}}$ in Eq. (i), we get

$$y = \sqrt{b^2 - \frac{b^2}{2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}}$$

$$\therefore \quad x = y = \frac{b}{\sqrt{2}} \Rightarrow 2x = 2y = \sqrt{2}b$$

Hence, area of rectangle is maximum, when $2x = 2y$, i.e. when rectangle is a square. **(1)**

22. Show that of all the rectangles with a given perimeter, the square has the largest area.

Delhi 2011

Using the formula perimeter of rectangle, $P = 2(x + y)$ and area of rectangle, $A = xy$. Making a relation between A and P, differentiate A with respect to x and simplify it.

Let x and y be the lengths of two sides of a rectangle. Again, let P denotes its perimeter and A be the area of rectangle. Given,

$$P = 2(x + y) \quad \textbf{(1)}$$

$$[\because \text{perimeter of rectangle} = 2(l + b)]$$

$$\Rightarrow \quad P = 2x + 2y$$

$$\Rightarrow \quad y = \frac{P - 2x}{2} \quad \text{...(i)} \ \textbf{(1)}$$

Also, we know that area of rectangle is given by

$$A = xy$$

$$\Rightarrow \quad A = x\left(\frac{P - 2x}{2}\right) \quad [\text{by using Eq. (i)}]$$

$$\Rightarrow \quad A = \frac{Px - 2x^2}{2} \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = \frac{P - 4x}{2} \quad \textbf{(1)}$$

Now, for maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{P - 4x}{2} = 0 \Rightarrow P = 4x \quad \textbf{(1)}$$

$$\Rightarrow \quad 2x + 2y = 4x \quad [\because P = 2x + 2y]$$

$$\Rightarrow \quad x = y$$

So, the rectangle is a square.

Also, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P - 4x}{2}\right) = -\frac{4}{2} = -2 < 0$

$\Rightarrow$ A is maximum.

Hence, area is maximum, when rectangle is a square. **(1)**

23. Show that of all the rectangles of given area, the square has the smallest perimeter.

Delhi 2011

Let x and y be the lengths of sides of a rectangle. A denotes its area and P be the perimeter.

Now, $A = xy$

[$\because$ area of rectangle $= l \times b$]

$\Rightarrow \quad y = \frac{A}{x}$ ...(i) **(1)**

And, $\quad P = 2(x + y)$

[$\because$ perimeter of rectangle $= 2(l + b)$]

$\Rightarrow \quad P = 2\left(x + \frac{A}{x}\right) \left[\because y = \frac{A}{x} \text{ by Eq.(i)}\right]$ **(1)**

On differentiating w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$ **(1)**

For maxima and minima, put $\frac{dP}{dx} = 0$

$\Rightarrow \quad 2\left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow 1 = \frac{A}{x^2}$ **(1)**

$\Rightarrow \quad A = x^2$

$\Rightarrow \quad xy = x^2$ [$\because A = xy$]

$\Rightarrow \quad x = y$ **(1)**

Also, $\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2\left(1 - \frac{A}{x^2}\right)\right] = 2\left(\frac{2A}{x^3}\right) = \frac{4A}{x^3} > 0$

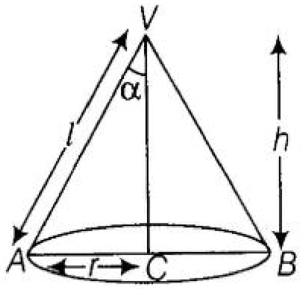
Here, x and A being the side and area of rectangle can never be negative. So, P is minimum.

Hence, perimeter of rectangle is minimum, when rectangle is a square. **(1)**

24. Show that the semi-vertical angle of a right circular cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

All India 2011, 2008; Delhi 2008C

Let h be the height, l be the slant height, r be the radius of base of the right circular cone and α be the semi-vertical angle of the cone.



In ΔVAC, by Pythagoras theorem, we have

$$l^2 = r^2 + h^2 \quad \Rightarrow \quad r^2 = l^2 - h^2 \qquad \text{...(i)} \ \textbf{(1)}$$

Let V be the volume of cone which is given by

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \quad V = \frac{\pi}{3}(l^2 - h^2) \cdot h$$

$$\Rightarrow \qquad V = \frac{\pi}{3}(l^2 h - h^3) \qquad \textbf{(1)}$$

On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{3}(l^2 - 3h^2) \qquad \textbf{(1)}$$

For maxima and minima, put $\frac{dV}{dh} = 0$

$$\Rightarrow \qquad \frac{\pi}{3}(l^2 - 3h^2) = 0 \ \Rightarrow l^2 = 3h^2$$

$$\Rightarrow \qquad r^2 + h^2 = 3h^2 \qquad [\because l^2 = r^2 + h^2]$$

$$\Rightarrow \qquad 2h^2 = r^2 \Rightarrow r = \sqrt{2}h \qquad \text{...(ii)} \ \textbf{(1)}$$

Now, in right angled ΔCVA, we have

$$\tan\alpha = \frac{AC}{VC}$$

$$\Rightarrow \qquad \tan\alpha = \frac{r}{h} \qquad [\because AC = r \text{ and } VC = h]$$

$$\Rightarrow \qquad \tan\alpha = \frac{\sqrt{2}h}{h} \quad [\because r = \sqrt{2}h, \text{ by Eq. (ii)}]$$

$$\Rightarrow \qquad \tan \alpha = \sqrt{2} \qquad \textbf{(1)}$$

$$\Rightarrow \qquad \alpha = \tan^{-1} \sqrt{2}$$

Also, $$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{\pi}{3}(l^2 - 3h^2)\right]$$

$$= \frac{\pi}{3}(-6h) = -2\pi h < 0 \text{ as } h > 0.$$

$\therefore V$ is maximum.

Hence, the volume is maximum, when

$$\alpha = \tan^{-1} \sqrt{2} \qquad \textbf{(1)}$$

25. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point $(1, 4)$.

HOTS; All India 2011, 2009C

Firstly, consider any point on the curve, use the formula of distance between two points. Then, square both sides and eliminate one variable with the help of given equation. Further, differentiate and solve it to get the result.

The given equation of curve is $y^2 = 2x$ and the given point is Q (1, 4).

Let $P(x, y)$ be the point, which is at a minimum distance from point Q (1, 4). **(1)**

Now, distance between points P and Q is given by

$$PQ = \sqrt{(1-x)^2 + (4-y)^2}$$

[using distance formula

$$S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}]$$

$$\Rightarrow \quad PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y}$$

$$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$$

On squaring both sides, we get

$$PQ^2 = x^2 + y^2 - 2x - 8y + 17$$

$$\Rightarrow \quad PQ^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17$$

$$\left[\because y^2 = 2x \text{ is given} \Rightarrow x = \frac{y^2}{2}\right]$$

$$\therefore \quad PQ^2 = \frac{y^4}{4} + y^2 - y^2 - 8y + 17$$

$$\Rightarrow \quad PQ^2 = \frac{y^4}{4} - 8y + 17 \qquad \textbf{(1)}$$

Let $PQ^2 = Z$

Then, $Z = \frac{y^4}{4} - 8y + 17$

On differentiating w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8 \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dZ}{dy} = 0$

$\Rightarrow \quad y^3 - 8 = 0$

$\Rightarrow \quad y^3 = 8$

$\Rightarrow \quad y = 2$ **(1)**

Also, $\frac{d^2Z}{dy^2} = \frac{d}{dy}(y^3 - 8) = 3y^2$

On putting $y = 2$, we get

$$\left[\frac{d^2Z}{dy^2}\right]_{y=2} = 3(2)^2 = 12 > 0$$

$\because \quad \frac{d^2Z}{dy^2} > 0$

$\therefore$ Z is minimum and therefore PQ is also minimum as $Z = PQ^2$. **(1)**

On putting $y = 2$ in the given equation, i.e. $y^2 = 2x$, we get

$$(2)^2 = 2x \Rightarrow 4 = 2x \Rightarrow x = 2$$

Hence, the point which is at a minimum distance from point (1, 4) is P (2, 2). **(1)**

26. A wire of length 28 m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the lengths of two pieces, so that the combined area of circle and square is minimum? **All India 2010**

Firstly, find length of circular part and its circumference and calculate the length of square part and its perimeter. Add these two terms and equate it to 28 m and apply second derivative test to get desired result.

Let x m be the side of the square and r be the radius of circular part. Then,

Length of square part = Perimeter of square

$$= 4 \times \text{Side} = 4x$$

and length of circular part

$$= \text{Circumference of circle} = 2\pi r$$

Given, length of wire = 28 $\Rightarrow 4x + 2\pi r = 28$

$$\Rightarrow \quad 2x + \pi r = 14$$

$$\therefore \quad x = \frac{14 - \pi r}{2} \qquad \text{...(i)} \ \mathbf{(1)}$$

Let A denotes the combined area of circle and square.

Then, $A = \pi r^2 + x^2$

$$\Rightarrow \quad A = \pi r^2 + \left(\frac{14 - \pi r}{2}\right)^2$$

$$\left[\because x = \frac{14 - \pi r}{2}, \text{ from Eq. (i)}\right]$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = 2\pi r + 2\left(\frac{14 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) \qquad \mathbf{(1)}$$

For maxima and minima, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) = 0$$

$$\Rightarrow \quad 2\pi r = \frac{14\pi - \pi^2 r}{}$$

2

$$\Rightarrow \qquad r = \frac{14}{\pi + 4} \qquad \textbf{(1)}$$

Also, $$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right)$$

$$= \frac{d}{dr}\left[2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right)\right]$$

$$\Rightarrow \qquad \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

Thus, $\frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum. **(1)**

Now, on putting $r = \frac{14}{\pi + 4}$ in Eq. (i), we get

$$x = \frac{14 - \pi\left(\frac{14}{\pi + 4}\right)}{2} = \frac{14\pi + 56 - 14\pi}{2(\pi + 4)} = \frac{28}{\pi + 4}$$

(1/2)

$$\therefore \qquad x = \frac{28}{\pi + 4} \quad \text{and} \quad r = \frac{14}{\pi + 4} \qquad \textbf{(1/2)}$$

Now, length of circular part

$$= 2\pi r = 2\pi \times \frac{14}{\pi + 4} = \frac{28\pi}{\pi + 4}$$

and length of square part $= 4x = 4 \times \frac{28}{\pi + 4}$

$$= \frac{112}{\pi + 4} \qquad \textbf{(1)}$$

which are the required length of two pieces.

27. An open tank with a square base and vertical sides is to be constructed from a metal sheet, so as to hold a given quantity of water. Show that the total surface area is least when depth of the tank is half its width. **All India 2010C**

Let the length, breadth and depth of the open tank be x, x and y, respectively. Length and breadth are same because given tank has a square base. Again, let V denotes its volume and S denotes its surface area. Now, given that

$$V = x^2 y \qquad \text{...(i)} \ \textbf{(1)}$$

Also, we know that the total surface area of the open tank is given by

$$S = x^2 + 4xy \qquad \text{...(ii)}$$

[$\because S$ = Area of square base + Area of the four walls]

On putting $y = \frac{V}{x^2}$ from Eq. (i) in Eq. (ii), we get

$$S = x^2 + 4x \cdot \frac{V}{x^2}$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x} \qquad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For maxima and minima, put $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 4V = 2x^3 \qquad \textbf{(1)}$$

$$\Rightarrow \quad 4x^2 y = 2x^3 \ [\because V = x^2 y, \text{ from Eq. (i)}]$$

$$\Rightarrow \quad 2y = x \Rightarrow y = \frac{x}{2}$$

So, depth of tank is half of its width. **(1)**

Also, $\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right) = \frac{d}{dx}\left(2x - \frac{4V}{x^2}\right)$

8V

$$= 2 + \frac{8v}{x^3}$$

$$= 2 + \frac{8 \cdot x^2 y}{x^3}$$ [from Eq. (i)]

$$= 2 + \frac{8y}{x} > 0 \text{ as } x > 0 \text{ and } y > 0$$ **(1)**

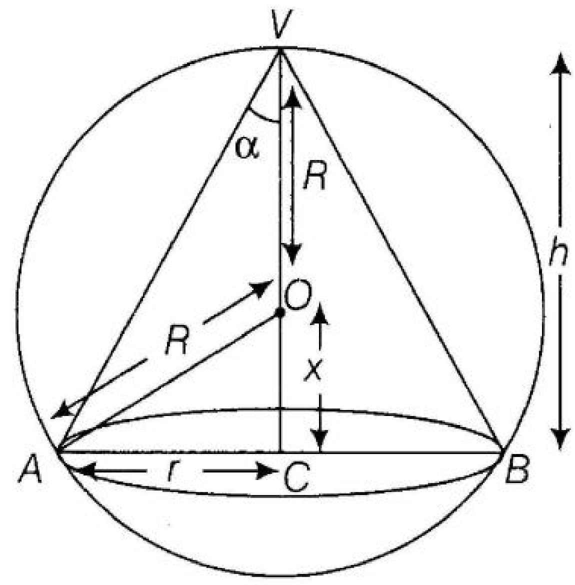
Thus, $\frac{d^2S}{dx^2} > 0,$

$\Rightarrow$ S is minimum.

Hence, total surface area of the tank is less when depth is half of its width. **(1)**

28. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere. **Delhi 2010C**

Let R be the radius of sphere, r be the radius of base of cone and h be the height of cone.



Then, from the figure,

$$h = R + x$$ ...(i)**(1/2)**

Let V denotes the volume of cone. Now, in right angled ΔOCA, we get

$$OA^2 = OC^2 + AC^2$$

[by Pythagoras theorem]

$\Rightarrow$ $R^2 = x^2 + r^2$

$\Rightarrow$ $r^2 = R^2 - x^2$ ...(ii) **(1)**

$\Rightarrow \quad r^2 = R^2 - x^2 \quad \ldots(ii) \; (1)$

Also, we know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi (R^2 - x^2)(R + x)$$

$$\left[\begin{array}{ll} \because & h = R + x, \text{ from Eq.(i)} \\ \text{and} & r^2 = R^2 - x^2, \text{ from Eq.(ii)} \end{array}\right]$$

$$\Rightarrow \quad V = \frac{\pi}{3}(R^3 + R^2 x - x^2 R - x^3)$$

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2xR - 3x^2)$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}[R^2 - 3xR + xR - 3x^2]$$

$$= \frac{\pi}{3}[R(R - 3x) + x(R - 3x)]$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R + x)(R - 3x) \quad \textbf{(1½)}$$

For maxima and minima, put $\frac{dV}{dx} = 0$

$$\Rightarrow \quad \frac{\pi}{3}(R + x)(R - 3x) = 0$$

$\Rightarrow$ Either $R + x = 0$ or $R - 3x = 0$

Now, $R + x = h$ which is height of cone. As h can never be zero. So, $R + x = 0$ is rejected.

$$\therefore \quad R - 3x = 0$$

$$\Rightarrow \quad 3x = R$$

$$\Rightarrow \quad x = \frac{R}{3} \quad \textbf{(1)}$$

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right)$$

$$= \frac{d}{dx}\left[\frac{\pi}{3}(R^2 - 2xR - 3x^2)\right]$$

$$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 6x)$$

$$\Rightarrow \quad \left[\frac{d^2V}{dx^2}\right]_{x=\frac{R}{3}} = \frac{\pi}{3}\left[-2R - \frac{6R}{3}\right]$$

$$= \frac{\pi}{3}(-4R) = -\frac{4\pi R}{3} < 0$$

Thus, $\frac{d^2V}{dx^2} < 0 \Rightarrow V$ is maximum. **(1)**

Now, volume of cone is $V = \frac{\pi}{3}(R^2 - x^2)(R + x)$

On putting $x = \frac{R}{3}$, we get

$$V = \frac{\pi}{3}\left[R^2 - \frac{R^2}{9}\right]\left[R + \frac{R}{3}\right]$$

$$= \frac{32\pi R^3}{81} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27}\text{[Volume of sphere]}$$

$$\left[\because \text{volume of sphere} = \frac{4}{3}\pi R^3\right]$$

Hence, volume of largest cone

$$= \frac{8}{27}\text{[Volume of sphere]}$$ **(1)**

29. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, with its vertex at one end of the major axis.

HOTS; Delhi 2010C

Given equation of ellipse is

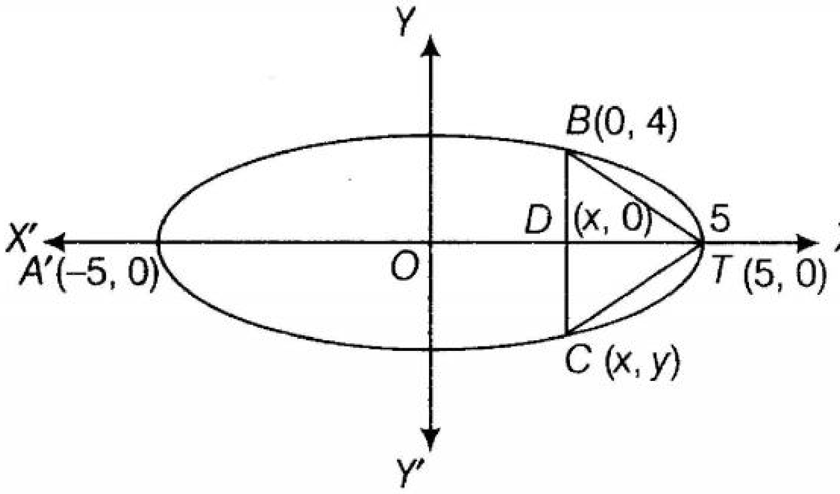
$$\frac{x^2}{25}+\frac{y^2}{16}=1.$$

Here, $a=5, b=4$

$\therefore$ $a>b$

So, major axis is along X-axis.

Let ΔBTC be the isosceles triangle which is inscribed in the ellipse. And $OD = x$, $BC = 2y$ and $TD = 5 - x$.



Let A denotes the area of triangle. Then, we have

$$A=\frac{1}{2}\times \text{Base}\times \text{Height}=\frac{1}{2}\times BC\times TD$$

$$\Rightarrow \quad A=\frac{1}{2}\cdot 2y(5-x)$$

$$\Rightarrow \quad A=y(5-x) \quad \textbf{(1)}$$

On squaring both sides, we get

$$A^2=y^2(5-x)^2 \quad \ldots\text{(i)}$$

Now, $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1 \Rightarrow \dfrac{y^2}{16}=1-\dfrac{x^2}{25}$

$$\Rightarrow \quad y^2=\frac{16}{25}(25-x^2)$$

On putting value of y^2 in Eq. (i), we get

$$A^2=\frac{16}{25}(25-x^2)(5-x)^2$$

25

Let $A^2 = Z$

Then, $Z = \frac{16}{25}(25 - x^2)(5 - x)^2$ **(1)**

On differentiating w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{16}{25}[(25 - x^2)\,2\,(5 - x)(-1) + (5 - x)^2(-2x)]$$

$$= \frac{16}{25}(-2)(5 - x)^2(2x + 5)$$

$$= \frac{-32}{25}(5 - x)^2(2x + 5)$$ **(1)**

For maxima and minima, put $\frac{dZ}{dx} = 0$

$\Rightarrow -\frac{32}{25}(5 - x)^2(2x + 5) = 0 \Rightarrow x = 5 \text{ or } -\frac{5}{2}$ **(1)**

Now, when $x = 5$, then

$$Z = \frac{16}{25}(25 - 25)(5 - 5)^2 = 0$$

which is not possible. So, $x = 5$ is rejected.

$$\therefore \qquad x = -\frac{5}{2}$$

Now, $\dfrac{d^2Z}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dZ}{dx}\right) = \dfrac{d}{dx}$

$$\left[-\frac{32}{25}(5-x)^2(2x+5)\right]$$

$$= -\frac{32}{25}[(5-x)^2 \cdot 2 - (2x+5)\,2\,(5-x)]$$

$$= -\frac{64}{25}(5-x)(-3x) = \frac{192x}{25}(5-x)$$

At $x = \dfrac{-5}{2}$, $\left[\dfrac{d^2Z}{dx^2}\right]_{x=-\frac{5}{2}} < 0$

$\Rightarrow Z$ is maximum. **(1)**

$\therefore$ Area A is maximum, when $x = -\dfrac{5}{2}$ and $y = 12$.

Also, the maximum area

$$Z = A^2 = \frac{16}{25}\left(25 - \frac{25}{4}\right)\left[5 + \frac{5}{2}\right]^2$$

$$= \frac{16}{25} \times \frac{75}{4} \times \frac{225}{4} = 3 \times 225$$

Hence, the maximum area, $A = \sqrt{3 \times 225}$

$= 15\sqrt{3}$ sq units **(1)**

NOTE If A is maximum/minimum, then A^2 is maximum/minimum.

30. Show that the right circular cylinder, open at the top and of given surface area and maximum volume is such that its height is equal to the radius of the base.

Delhi 2010; 2009C

Let V be the volume, S be the total surface area of a right circular cylinder which is open at the top. Again, let r be the radius of base and h be the height.

Now, $S = 2\pi rh + \pi r^2$

[$\because$ cylinder is open at top]

$$\Rightarrow \quad h = \frac{S - \pi r^2}{2\pi r} \qquad \text{...(i)} \ \mathbf{(1)}$$

Also, volume of cylinder is given by

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{rS - \pi r^3}{2} \qquad \mathbf{(1)}$$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2} \qquad \mathbf{(1)}$$

For maxima and minima, put $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{S - 3\pi r^2}{2} = 0 \ \Rightarrow \ S = 3\pi r^2$$

$$\Rightarrow \quad 2\pi rh + \pi r^2 = 3\pi r^2 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad h = r \qquad \mathbf{(1)}$$

$\therefore$ Height of cylinder = Radius of the base

Also, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 3\pi r^2}{2}\right) = -\frac{6\pi r}{2}$

$$= -3\pi r < 0, \text{ as } r > 0 \qquad \mathbf{(1)}$$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

Hence, volume of cylinder is maximum, when its height is equal to radius of the base. **(1)**

31. A manufacturer can sell x items at a price of ₹$\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹$\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to reach maximum profit.

HOTS; All India 2009

Given the manufacturer sells x items at price of ₹$\left(5 - \frac{x}{100}\right)$ each.

∴ Total revenue obtained

$$= ₹\left[x\left(5 - \frac{x}{100}\right)\right] = ₹\left(5x - \frac{x^2}{100}\right) \quad \textbf{(1)}$$

Also, cost price of x items = ₹$\left(\frac{x}{5} + 500\right)$

Let $P(x)$ be the profit function. Then, we know that

Profit = Revenue – Cost **(1)**

$$\therefore \quad P = \left(5x - \frac{x^2}{100}\right) - \left(\frac{x}{5} + 500\right)$$

$$\Rightarrow \quad P = \frac{-x^2}{100} + \frac{24x}{5} - 500 \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dP}{dx} = \frac{-2x}{100} + \frac{24}{5}$$

For maxima and minima, put $\frac{dP}{dx} = 0$ **(1)**

$$\Rightarrow \quad \frac{-2x}{100} + \frac{24}{5} = 0$$

$$\Rightarrow \quad x = 240 \quad \textbf{(1)}$$

Also, $$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(-\frac{2x}{100} + \frac{24}{5}\right)$$

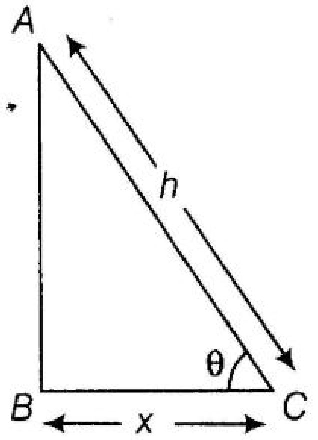
$$= -\frac{2}{100} = -\frac{1}{50} < 0$$

Thus, at $x = 240$, $\frac{d^2P}{dx^2} < 0 \Rightarrow P$ is maximum.

Hence, number of items sold to have maximum profit is 240. **(1)**

32. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\pi/3$. HOTS; All India 2009

Let ABC be the right angled triangle with $BC = x$ and $AC = h$.



Now, given that $h + x = a$ ...(i)

where, a = constant **(1)**

And A denotes the area of triangle. Then,

$$A = \frac{1}{2} \times BC \times AB$$

$$\Rightarrow \quad A = \frac{1}{2} x \cdot \sqrt{h^2 - x^2}$$

$$\begin{bmatrix} \text{in right angled } \Delta ABC, \\ AB^2 = AC^2 - BC^2 = h^2 - x^2 \\ \therefore AB = \sqrt{h^2 - x^2} \end{bmatrix} \quad \textbf{(1)}$$

On squaring both sides, we get

$$A^2 = \frac{x^2}{4}(h^2 - x^2) \Rightarrow A^2 = \frac{x^2}{4}[(a - x)^2 - x^2]$$

$$[\because h = a - x, \text{ from Eq. (i)}]$$

$$\Rightarrow \quad A^2 = \frac{a^2x^2 - 2ax^3}{4} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$2A \frac{dA}{dx} = \frac{1}{4}(2a^2x - 6ax^2) \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{8A}(2a^2x - 6ax^2)$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{1}{8A}(2a^2x - 6ax^2) = 0$$

$$\Rightarrow \quad 2a^2x = 6ax^2 \Rightarrow x = \frac{a}{3} \qquad \textbf{(1/2)}$$

Again, differentiating Eq. (ii) w.r.t. x, we get

$$2A \cdot \frac{d^2A}{dx^2} + \frac{dA}{dx} \cdot 2\frac{dA}{dx} = \frac{1}{4}(2a^2 - 12ax)$$

$$\Rightarrow \quad 2A \cdot \frac{d^2A}{dx^2} + 2\left(\frac{dA}{dx}\right)^2 = \frac{1}{4}(2a^2 - 12ax)$$

On putting $\frac{dA}{dx} = 0$ and $x = \frac{a}{3}$, we get

$$2A\frac{d^2A}{dx^2} = \frac{1}{4}\left[2a^2 - 12a \times \frac{a}{3}\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{1}{8A}[2a^2 - 4a^2]$$

$$= -\frac{2a^2}{8A} = -\frac{a^2}{4A} < 0$$

$$\because \quad \frac{d^2A}{dx^2} < 0 \Rightarrow A \text{ is maximum.} \qquad \textbf{(1½)}$$

Also, in the given right angled ΔABC, we have

$$\cos\theta = \frac{BC}{AC} = \frac{x}{h} = \frac{x}{a - x} \qquad [\because h = a - x]$$

$$\therefore \quad \cos\theta = \frac{\left(\frac{a}{3}\right)}{\left(a - \frac{a}{3}\right)} = \frac{\left(\frac{a}{3}\right)}{\left(\frac{2a}{3}\right)} = \frac{a}{3} \times \frac{3}{2a} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$$

Hence, area of triangle is maximum, when $\theta = \frac{\pi}{3}$. **(1)**

33. A tank with rectangular base and rectangular sides, open at the top is to be constructed, so that its depth is 2 m and volume is 8 m^3.
If building of tank cost ₹ 70 per sq m for the base and ₹ 45 per sq m for sides. What is the cost of least expensive tank?

HOTS; Delhi 2009

Let x m be the length, y m be the breadth and $h = 2$ m be the depth of the tank. Let ₹ H be the total cost for building the tank. Now, given that $h = 2$ m and volume of tank $= 8 \text{ m}^3$

Also, area of the rectangular base of the tank

$= \text{Length} \times \text{Breadth} = xy \text{ m}^2$ **(1)**

and the area of the four rectangular sides

$= 2\,(\text{Length} + \text{Breadth}) \times \text{Height}$

$= 2\,(x + y) \times 2 = 4\,(x + y) \text{ m}^2$ **(1)**

$\therefore$ Total cost, $H = 70 \times xy + 45 \times 4\,(x + y)$

$\Rightarrow \qquad H = 70xy + 180\,(x + y)$ ...(i)

Also, volume of tank $= 8 \text{ m}^3$

$\Rightarrow \quad l \times b \times h = 8 \quad \Rightarrow x \times y \times 2 = 8$

$$\Rightarrow \qquad y = \frac{4}{x} \qquad \text{...(ii)} \ \mathbf{(1)}$$

On putting the value of y from Eq. (ii) in Eq. (i), we get

$$H = 70x \times \frac{4}{x} + 180\left(x + \frac{4}{x}\right)$$

$$\Rightarrow \qquad H = 280 + 180\left(x + \frac{4}{x}\right) \qquad \text{...(iii)}$$

On differentiating w.r.t. x, we get

$$\frac{dH}{dx} = 180\left(1 - \frac{4}{x^2}\right)$$

For maxima and minima, put $\frac{dH}{dx} = 0$

$$\Rightarrow 180\left(1 - \frac{4}{x^2}\right) = 0 \Rightarrow 1 - \frac{4}{x^2} = 0$$

$$\Rightarrow \qquad \frac{4}{x^2} = 1 \Rightarrow x^2 = 4$$

$$\Rightarrow \qquad x = 2 \qquad [\because x > 0] \ \mathbf{(1)}$$

Also, $$\frac{d^2H}{dx^2} = \frac{d}{dx}\left(\frac{dH}{dx}\right) = \frac{d}{dx}\left[180\left(1 - \frac{4}{x^2}\right)\right]$$

$$= \frac{8}{x^3} \times 180$$

At $x = 2$, $\left[\frac{d^2H}{dx^2}\right]_{x=2} = \frac{8}{2^3} \times 180 = 180 > 0$

$\because \qquad \frac{d^2H}{dx^2} > 0 \Rightarrow H$ is least at $x = 2$. **(1)**

Also, the least cost $= 280 + 180\left(2 + \frac{4}{2}\right)$

[put $x = 2$ in Eq. (iii) to get least cost H]

$= 280 + 180 \times 4 = 280 + 720 = ₹1000$

Hence, the cost of least expensive tank is ₹1000. **(1)**

34. Show that the height of the closed right circular cylinder, of given volume and minimum total surface area, is equal to its diameter. **All India 2008C**

> Here, we have two independent variables r and h, so we eliminate one variable. For this, find the value of h in terms of r and V and put in surface area, then use the second derivative test.

Let r be the radius of base, h be the height, V be the volume and S be the total surface area of the closed right circular cylinder. Then, given

$$V = \pi r^2 h$$

$$\Rightarrow \quad h = \frac{V}{\pi r^2} \quad \ldots(i) \ \mathbf{(1)}$$

Now, we know that, total surface area of cylinder is given by

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$\left[\because h = \frac{V}{\pi r^2}, \text{ from Eq.(i)}\right]$$

$$\Rightarrow \quad S = 2\pi r^2 + \frac{2V}{r}$$

On differentiating w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} \quad \mathbf{(1½)}$$

For maxima and minima, put $\frac{dS}{dr} = 0$

$$\Rightarrow \quad 4\pi r - \frac{2V}{r^2} = 0 \quad \Rightarrow \quad V = 2\pi r^3$$

$$\Rightarrow \quad \pi r^2 h = 2\pi r^3 \quad [\because V = \pi r^2 h]$$

$$\Rightarrow \quad h = 2r$$

i.e. Height = Diameter of the base **(1½)**

Also, $\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(4\pi r - \frac{2V}{r^2}\right)$

$$= 4\pi + \frac{4V}{r^3} > 0, \text{ as } r > 0 \text{ and } V > 0$$

Thus, $\frac{d^2S}{dr^2} > 0$

$\Rightarrow S$ is minimum. **(1)**

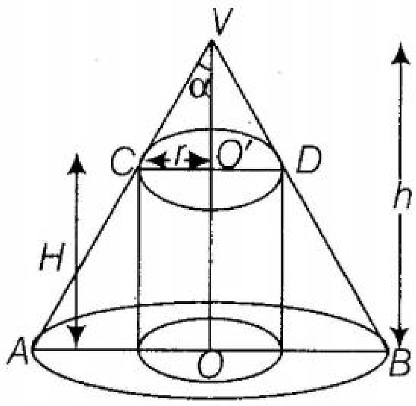
Hence, total surface area S is minimum, when height is equal to the diameter of the base. **(1)**

35. Show that the volume of the greatest cylinder can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

All India 2008

Let VAB be the given cone of height h and semi-vertical angle α. Again, let V denotes the volume of the cylinder. From the figure, we have

(1/2)



Radius of base of cylinder $= O'C = r$

$H =$ Height of cylinder $= OO' = h - VO'$

Now, in right angled $\Delta VO'C$, we have

$$\tan \alpha = \frac{O'C}{VO'} = \frac{r}{VO'} \quad [\because O'C = r]$$

$$\Rightarrow \quad VO' = \frac{r}{\tan \alpha}$$

$$\Rightarrow \quad VO' = r \cot \alpha \qquad \textbf{(1)}$$

$\therefore$ Height of cylinder, $H = OO'$

$$= h - VO' = h - r \cot \alpha$$

Now, volume of cylinder is given by

$$V = \pi r^2 H$$

$$\Rightarrow \quad V = \pi r^2 (h - r \cot \alpha) \qquad \ldots(i)$$

$$[\because H = h - r \cot \alpha]$$

$$\Rightarrow \quad V = \pi r^2 h - \pi r^3 \cot \alpha$$

On differentiating w.r.t.r, we get

$$\frac{dV}{dr} = 2\pi rh - 3\pi r^2 \cot \alpha \qquad \textbf{(1½)}$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$\Rightarrow 2\pi rh - 3\pi r^2 \cot\alpha = 0$

$$\Rightarrow \qquad r = \frac{2h}{3} \tan\alpha \qquad \textbf{(1)}$$

Also,
$$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$$
$$= \frac{d}{dr}(2\pi rh - 3\pi r^2 \cot\alpha)$$
$$\Rightarrow \qquad \frac{d^2V}{dr^2} = 2\pi h - 6\pi r \cot\alpha$$

At $r = \frac{2h}{3}$, $\left[\frac{d^2V}{dr^2}\right]_{r = \frac{2h}{3}\tan\alpha}$

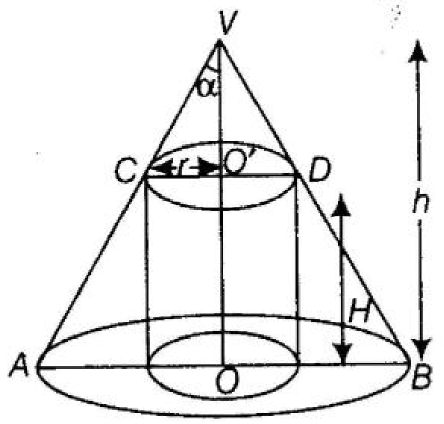
$$= 2\pi h - 6\pi \cot\alpha \left(\frac{2h}{3}\tan\alpha\right)$$
$$= 2\pi h - 4\pi h \tan\alpha \cot\alpha$$
$$= 2\pi h - 4\pi h$$
$$[\because \tan\alpha \cot\alpha = 1]$$
$$= -2\pi h < 0 \text{ as } h > 0 \qquad \textbf{(1)}$$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

36. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

Delhi 2008

Let VAB be the given cone of height h and a semi-vertical angle α. Again let V denotes the volume of cylinder. From the figure, we have



(1/2)

H = Height of cylinder $= OO' = h - VO'$

Now, in right angled $\Delta VO'C$, we get

$$\tan\alpha = \frac{O'C}{VO'} = \frac{r}{VO'}$$

$\Rightarrow \qquad VO' = \dfrac{r}{\tan\alpha} = r\cot\alpha$ **(1)**

$\therefore$ Height of cylinder $= H$

$$= h - VO' = h - r\cot\alpha$$

Also, radius of base of cylinder $= O'C = r$

$\therefore$ Volume of cylinder is given by

$$V = \pi r^2 H \Rightarrow V = \pi r^2 (h - r\cot\alpha)$$

$$[\because H = h - r\cot\alpha]$$

$\Rightarrow \qquad V = \pi r^2 h - \pi r^3 \cot\alpha$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = 2\pi rh - 3\pi r^2 \cot\alpha$$

For maxima and minima, put $\dfrac{dV}{dr} = 0$

$\Rightarrow \qquad 2\pi rh - 3\pi r^2 \cot\alpha = 0$

$\Rightarrow \qquad r = \dfrac{2h}{3}\tan\alpha$ **(1½)**

Now, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$

$= \frac{d}{dr}(2\pi rh - 3\pi r^2 \cot\alpha)$

$= 2\pi h - 6\pi r \cot\alpha$

At $r = \frac{2h}{3}\tan\alpha$, $\left[\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\alpha}$

$= 2\pi h - 6\pi \cot\alpha \cdot \frac{2h}{3}\tan\alpha$

$= 2\pi h - 4\pi h \tan\alpha \cot\alpha$

$= 2\pi h - 4\pi h$ $[\because \tan\alpha \cot\alpha = 1]$

$= -2\pi h < 0$ as $h > 0$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow$ Volume is maximum. **(1½)**

Now, height of cylinder, $H = h - r\cot\alpha$

$= h - \frac{2h}{3}\tan\alpha \cot\alpha$ $\left[\because r = \frac{2h}{3}\tan\alpha\right]$

$= h - \frac{2h}{3} = \frac{h}{3}$ $[\because \tan\alpha \cdot \cot\alpha = 1]$

Hence, height of cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$. **(1½)**